Varianta 003

SUBIECTUL **I**

a) $2i$.
b) $4\sqrt{2}$.
c) 0 .
d) -1.
e) 4 **.**
f) 1.

SUBIECTUL II

1.
a) $\hat{2}$.
b) 48 .
c) 81.
d) $S = \{0,2\}$.
e) $\frac{2}{5}$.

2.
a) $2^x \ln 2$.
b) $\frac{1}{\ln 2} + 1$.
c) $\ln 2$.
d) $f'(x) > 0, \forall\ x \in \mathbf{R}$.,deci funcția f este strict crescătoare pe $\mathbf{R}$.
e) $\frac{25}{2}$.

SUBIECTUL III

a) $A + I_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = B$
b) $\det A = 0$ **;** rang $A = 1$.
c) $A^2 = A \cdot A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A$.

d) Din **c)**,avem $A^2 = A$.
Presupunem $A^k = A$ şi demonstrăm $A^{k+1} = A, \forall k \geq 1. k \in \mathbf{N}$ **.**
$A^{k+1} = A^k \cdot A = A \cdot A = A^2 = A$ .Deci $A^n = A, \forall n \geq 1., n \in \mathbf{N}$ **.** Atunci $A^{2007} = A$.
e) Pentru $n = 1$ se obține $B = A + I_2$,evident din **a)** , deci s-a realizat etapa de verificare.
Presupunem $B^k = I_2 + (2^k - 1)A, \forall\ k \in \mathbf{N}^*$ şi demonstrăm
$B^{k+1} = I_2 + (2^{k+1} - 1)A,\ \forall\ k \in \mathbf{N}^*$ **.**

$B^{k+1}=B^k\cdot B=\left[I_2+\left(2^k-1\right)A\right]\left(A+I_2\right)=A+I_2+\left(2^k-1\right)A^2+\left(2^k-1\right)A=A+I_2+\left(2^k-1\right)A+\left(2^k-1\right)A=$

$=A\left(2^{k+1}-1\right)+I_2=I_2+\left(2^{k+1}-1\right)A$, adevarată $\forall\ k\in\ \mathbf{N}^*$.

Deci $B^n=I_2+\left(2^n-1\right)A$, $\forall n\in\ \mathbf{N}^*$.

f) $aA+bB+cI_2=\begin{pmatrix} a+2b+c & a+b \\ 0 & b+c\end{pmatrix}\neq\begin{pmatrix} 5 & 6 \\ 7 & 8\end{pmatrix}$ deoarece $0\neq 7$.

g) Matricea $X=A^n+B^n$ este inversabilă, dacă $\det\left(A^n+B^n\right)\neq 0$, $\forall\ n\in\mathbf{N}^*$.

Dar din **d)**,avem $A^n=A$,iar din **e),** avem $B^n=I_2+\left(2^n-1\right)A$.

Obținem $A^n+B^n=\begin{pmatrix} 1 & 1 \\ 0 & 0\end{pmatrix}+\begin{pmatrix} 2^n & 2^n \\ 0 & 0\end{pmatrix}=\begin{pmatrix} 2^n+1 & 2^n \\ 0 & 0\end{pmatrix}$.

Obținem $\det\left(A^n+B^n\right)=2^n+1\neq 0$, $\forall\ n\in\mathbf{N}^*$,deci $X=A^n+B^n$ este inversabilă, $\forall\ n\in\mathbf{N}^*$.

SUBIECTUL IV

a) $f'(x)=-\sin x$;

$g'(x)=-\dfrac{1}{x^2}$.

b) $\int_{\frac{\pi}{2}}^{\pi} f^2(x)dx=\int_{\frac{\pi}{2}}^{\pi}\cos^2 dx=\int_{\frac{\pi}{2}}^{\pi}\frac{1+\cos 2x}{2}dx=\left.\frac{x}{2}\right|_{\frac{\pi}{2}}^{\pi}+\frac{1}{4}\int_{\frac{\pi}{2}}^{\pi}\left(\sin 2x\right)'dx=\frac{\pi}{4}+\frac{1}{4}\left.\left(\sin 2x\right)\right|_{\frac{\pi}{2}}^{\pi}=\frac{\pi}{4}$.

c) $\int_{\frac{\pi}{2}}^{\pi} g^2(x)dx=\int_{\frac{\pi}{2}}^{\pi}\frac{1}{x^2}dx=\left.\left(-\frac{1}{x}\right)\right|_{\frac{\pi}{2}}^{\pi}=\frac{1}{\pi}$.

d) $\lim\limits_{\substack{x\to 0\\ x>0}} g(x)=\lim\limits_{\substack{x\to 0\\ x>0}}\frac{1}{x}=+\infty$ deci $x=0$ este asimptotă verticală .

e) $t^2\cos^2 x-2t\dfrac{\cos x}{x}+\dfrac{1}{x^2}\geq 0$,devine $\left(t\cos x-\dfrac{1}{x}\right)^2\geq 0$, adevărată $\forall t\in\mathbf{R}$, $\forall\ x>0$.

f) Integrând inegalitatea de la punctul **e)**, obținem

$t^2\int_{\frac{\pi}{2}}^{\pi}\cos^2 xdx-2t\int_{\frac{\pi}{2}}^{\pi}\frac{\cos x}{x}dx+\int_{\frac{\pi}{2}}^{\pi}\frac{1}{x^2}dx\geq 0$, $\forall t\in\mathbf{R}.$

g) Din **f)**,avem că $\Delta\leq 0$; atunci $\Delta=\left(2\int_{\frac{\pi}{2}}^{\pi}\frac{\cos x}{x}dx\right)^2-4\int_{\frac{\pi}{2}}^{\pi}\cos^2 xdx\cdot\int_{\frac{\pi}{2}}^{\pi}\frac{1}{x^2}dx\leq 0$ sau

$$\left(\int_{\frac{\pi}{2}}^{\pi} \frac{\cos x}{x}\,dx\right)^2 \le \left(\int_{\frac{\pi}{2}}^{\pi} \cos^2 x dx\right) \cdot \left(\int_{\frac{\pi}{2}}^{\pi} \frac{1}{x^2}\,dx\right).$$