Soluţii

1.a) Fie $X=\begin{pmatrix} m & n \\ p & q \end{pmatrix} \in M_2(\mathbb{R})$ o matrice astfel incat AX=XA.

$$A\cdot X=\begin{pmatrix} a & b \\ b & a \end{pmatrix}\cdot\begin{pmatrix} m & n \\ p & q \end{pmatrix}=\begin{pmatrix} am+bp & an+bq \\ bm+ap & bn+aq \end{pmatrix}$$

$$X\cdot A=\begin{pmatrix} m & n \\ p & q \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & a \end{pmatrix}=\begin{pmatrix} ma+nb & mb+na \\ pa+qb & pb+qa \end{pmatrix}$$

Din egalitatea AX=XA rezultă sistemul $\begin{cases} am+bp=ma+nb \\ an+bq=mb+na \\ bm+ap=pa+qb \\ bn+aq=pb+qa \end{cases} \Rightarrow \begin{cases} p=n\overset{not.}{=}v \\ q=m\overset{not.}{=}u \end{cases}$

Deci $X=\begin{pmatrix} u & v \\ v & u \end{pmatrix}$.

b) Demonstrăm prin inducţie matematică după $n\in\mathbb{N}^*$.

Notăm $P(n): A^n=\begin{pmatrix} x_n & y_n \\ y_n & x_n \end{pmatrix}, x_n=\dfrac{(a+b)^n+(a-b)^n}{2}, y_n=\dfrac{(a+b)^n-(a-b)^n}{2}, n\ge 1$

<u>Etapa verificării</u>:

Pentru n=1 avem $P(1): A^1=\begin{pmatrix} x_1 & y_1 \\ y_1 & x_1 \end{pmatrix}=\begin{pmatrix} a & b \\ b & a \end{pmatrix}, x_1=\dfrac{(a+b)+(a-b)}{2}=a, y_1=\dfrac{(a+b)-(a-b)}{2}=b$ adevărat.

<u>Etapa demonstraţiei</u>:

Presupunem P(k) adevărată şi demonstrăm că P(k+1) este adevărată.

$P(k): A^k=\begin{pmatrix} x_k & y_k \\ y_k & x_k \end{pmatrix}, x_k=\dfrac{(a+b)^k+(a-b)^k}{2}, y_k=\dfrac{(a+b)^k-(a-b)^k}{2}$ este presupusă adevărată(deci ne putem folosi de ea)

$P(k+1): A^{k+1}=\begin{pmatrix} x_{k+1} & y_{k+1} \\ y_{k+1} & x_{k+1} \end{pmatrix}, x_{k+1}=\dfrac{(a+b)^{k+1}+(a-b)^{k+1}}{2}, y_{k+1}=\dfrac{(a+b)^{k+1}-(a-b)^{k+1}}{2}$ trebuie demonstrată.

$$A^{k+1}=A^k\cdot A=\begin{pmatrix} \dfrac{(a+b)^k+(a-b)^k}{2} & \dfrac{(a+b)^k-(a-b)^k}{2} \\ \dfrac{(a+b)^k-(a-b)^k}{2} & \dfrac{(a+b)^k+(a-b)^k}{2} \end{pmatrix}\cdot\begin{pmatrix} a & b \\ b & a \end{pmatrix}=$$

$$=\begin{pmatrix} a\cdot\dfrac{(a+b)^k+(a-b)^k}{2}+b\cdot\dfrac{(a+b)^k-(a-b)^k}{2} & b\cdot\dfrac{(a+b)^k+(a-b)^k}{2}+a\cdot\dfrac{(a+b)^k-(a-b)^k}{2} \\ b\cdot\dfrac{(a+b)^k+(a-b)^k}{2}+a\cdot\dfrac{(a+b)^k-(a-b)^k}{2} & a\cdot\dfrac{(a+b)^k+(a-b)^k}{2}+b\cdot\dfrac{(a+b)^k-(a-b)^k}{2} \end{pmatrix}=$$

$$=\begin{pmatrix} \dfrac{(a+b)^{k+1}+(a-b)^{k+1}}{2} & \dfrac{(a+b)^{k+1}-(a-b)^{k+1}}{2} \\ \dfrac{(a+b)^{k+1}-(a-b)^{k+1}}{2} & \dfrac{(a+b)^{k+1}+(a-b)^{k+1}}{2} \end{pmatrix}=\begin{pmatrix} x_{k+1} & y_{k+1} \\ y_{k+1} & x_{k+1} \end{pmatrix}$$

unde am notat $x_{k+1}=\dfrac{(a+b)^{k+1}+(a-b)^{k+1}}{2}, y_{k+1}=\dfrac{(a+b)^{k+1}-(a-b)^{k+1}}{2}$ deci P(k+1) a fost demonstrată.

In concluzie P(n) este adevărată $\forall n\ge 1$.

c) Notăm $A=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$. Fie X o soluție a ecuației date.

$X\cdot X^3=X^3\cdot X\Rightarrow X\cdot A=A\cdot X\Rightarrow X=\begin{pmatrix}u & v\\ v & u\end{pmatrix}$ conform punctului **a)**.

$$X^3=\begin{pmatrix}\frac{(u+v)^3+(u-v)^3}{2} & \frac{(u+v)^3-(u-v)^3}{2}\\ \frac{(u+v)^3-(u-v)^3}{2} & \frac{(u+v)^3+(u-v)^3}{2}\end{pmatrix}=\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$ conform punctului **b)**

$$\Rightarrow\begin{cases}\frac{(u+v)^3+(u-v)^3}{2}=2\\ \frac{(u+v)^3-(u-v)^3}{2}=1\end{cases}\Rightarrow\begin{cases}(u+v)^3+(u-v)^3=4\\ (u+v)^3-(u-v)^3=2\end{cases}\Rightarrow\begin{cases}(u+v)^3=3\\ (u-v)^3=1\end{cases}\Rightarrow\begin{cases}u+v=\sqrt[3]{3}\\ u-v=1\end{cases}\Rightarrow\begin{cases}u=\frac{\sqrt[3]{3}+1}{2}\\ v=\frac{\sqrt[3]{3}-1}{2}\end{cases}$$

deci $X=\begin{pmatrix}\frac{\sqrt[3]{3}+1}{2} & \frac{\sqrt[3]{3}-1}{2}\\ \frac{\sqrt[3]{3}-1}{2} & \frac{\sqrt[3]{3}+1}{2}\end{pmatrix}$ este soluția ecuației date.

2.a) $\mathbb{Z}_7=\{\hat{0},\hat{1},\hat{2},...,\hat{6}\}$ este mulțime finită deci le putem verifica pe toate:

$(\hat{1})^6=\hat{1}$

$(\hat{2})^6=\hat{1}$

$(\hat{3})^6=\hat{1}$

$(\hat{4})^6=\hat{1}$

$(\hat{5})^6=\hat{1}$

$(\hat{6})^6=\hat{1}$

b) $(x^3-\hat{4})(x^3+\hat{4})=x^6-\hat{2}=x^6+\hat{5},\forall x\in\mathbb{Z}_7$

c) Fie $a\in\mathbb{Z}_7, a\neq\hat{0}$. Tripletul $(\mathbb{Z}_7,+,\cdot)$ este corp comutativ (deoarece 7 este prim) deci există $a^{-1}\in\mathbb{Z}_7, a^{-1}\neq\hat{0}$.

$f(a^{-1})=(a^{-1})^6+a\cdot a^{-1}+\hat{5}=\hat{1}+\hat{1}+\hat{5}=\hat{0}$ deci f este divizibil prin $X-a^{-1}$ adică f este reductibil in $\mathbb{Z}_7[X]$.

Pentru $a=\hat{0}$ avem $f=x^6+\hat{5}=(x^3-\hat{4})(x^3+\hat{4})$ deci f este reductibil şi in acest caz.