Soluții

1.a) $f'(x)=1-\cos x\geq 0$ rezultă că f este crescătoare (am folosit faptul că $\cos x\leq 1$).

b) $f(x_n)=n\Rightarrow x_n-\sin x_n=n\Rightarrow x_n=n+\sin x_n\geq n-1$ deci şirul $(x_n)_n$ este nemărginit.

c) $n-1\leq x_n\leq n+1\Rightarrow \frac{n-1}{n}\leq\frac{x_n}{n}\leq\frac{n+1}{n}\Rightarrow\lim_{n\to+\infty}\frac{x_n}{n}=1$ (am aplicat criteriul cleştelui).

$\frac{n-1}{n}\to 1$, $\frac{n+1}{n}\to 1$

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2.a) $\int_0^{\frac{1}{2}}\left[\frac{1}{1-x}-\frac{x^2}{1-x}\right]dx=\int_0^{\frac{1}{2}}\left[\frac{1-x^2}{1-x}\right]dx=\int_0^{\frac{1}{2}}\left[\frac{(1-x)(1+x)}{1-x}\right]dx=\int_0^{\frac{1}{2}}(1+x)dx=\left(x+\frac{x^2}{2}\right)\Big|_0^{\frac{1}{2}}=\frac{1}{2}+\frac{1}{8}=\frac{5}{8}$.

b) Studiem monotonia funcției g_n

$g_n'(x)=\frac{n\cdot x^{n-1}(1-x)+x^n}{(1-x)^2}>0$ deci funcția g_n este crescătoare

$\Rightarrow 0\leq g_n(x)\leq g_n\left(\frac{1}{2}\right),\forall x\in\left[0,\frac{1}{2}\right]$

$0\leq\int_0^{\frac{1}{2}}g_n(x)dx\leq g_n\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}1dx$

$g_n\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}1dx=\frac{\left(\frac{1}{2}\right)^n}{\frac{1}{2}}\cdot x\Big|_0^{\frac{1}{2}}=\frac{1}{2^{n-1}}\cdot\frac{1}{2}=\frac{1}{2^n},\forall n\in\mathbb{N}^*$ de unde rezultă concluzia.

c) $\int_0^{\frac{1}{2}}\frac{1-x^n}{1-x}dx=\int_0^{\frac{1}{2}}\left(1+x+x^2+...+x^{n-1}\right)dx=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+...+\frac{x^n}{n}\right)\Big|_0^{\frac{1}{2}}=\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+...+\frac{1}{n\cdot 2^n}$

$\Rightarrow\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+...+\frac{1}{n\cdot 2^n}=\int_0^{\frac{1}{2}}\frac{1}{1-x}dx-\int_0^{\frac{1}{2}}\frac{x^n}{1-x}dx=-\ln(1-x)\Big|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}g_n(x)dx=\ln 2-\int_0^{\frac{1}{2}}g_n(x)dx$

$\Rightarrow\lim_{n\to+\infty}\left(\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+...+\frac{1}{n\cdot 2^n}\right)=\ln 2-\underbrace{\lim_{n\to+\infty}\int_0^{\frac{1}{2}}g_n(x)dx}$

Conform punctului b)

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